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ZAK TRANSFORM FOR BOEHMIANS

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ABSTRACT

It is known that the classical Zak Transform is a linear unitary transformation from $L^2(R)$ onto L^2 (Q) whose image can be completely characterized. In this paper, we shall construct a Boehmian space B_1 containing $L^2(R)$ and another Boehmian space B_2 containing L^2 (Q) and define Zak transform as a continuous linear map of B_1 onto B_2. We shall also prove that this extended definition is consistent with the classical definition and that there are Boehmains which are not L^2 – functions but for which we can define the generalized Zak transform.

KEYWORDS: Boehmians, Convolution, Lebesgue Measurable Functions, Sequence, and Zak Transform

Mathematics Subject Classification: 46F30; 42A38; 65R10; 46T30; 46F99.

INTRODUCTION

Motivated by the concept of Boehme's regular operator [4] the theory of Boehmian spaces is developed in the literature [2], [5], [7], [8], and [11]. Further various integral transforms are also studied in the context of Boehmians with their properties [1], [6], [9], and [10].

In this paper we shall extend the concept of Zak transform in the context of Boehmians and study its properties. We shall recall the classical theory of Zak transform in section 2 and we construct suitable Boehmian spaces for our definition of Zak transform in section 3. In section 4 we define the Zak transform and study its properties. Let R, ₵ denote the usual real line, the complex plane respectively and Q = [0, 1) x [0, 1).

Let $L^2(R)$ and L^2 (Q) denote the set of all Lebesgue measurable functions f on R with $\int_{-\infty}^{\infty} |f(x)|^2 \, dx < \infty$ and the set of all Lebesgue measurable functions F on Q with the double integral $\int_0^1 \int_0^1 |F(t,w)|^2 \, dt \, dw < \infty$ respectively, where dt, dw denotes the Lebesque measures on [o, 1).

$$||f||_2^2 = \int_{-\infty}^{\infty} |f(x)|^2 \, dx \text{ and } ||F||_2^2 = \int_0^1 \int_0^1 |F(t,w)|^2 \, dt \, dw.$$

PRELIMINARIES

We first recall the theory of Zak transform from L^2 (R) onto L^2 (Q).

The Zak transform $Z_a[f]$ of a function f is defined by

$$\Phi(f) = Z_a[f](t,w) = (Z_a f)(t,w) \triangleq \sqrt{a} \sum_{k=-\infty}^{\infty} f(at + ak) \, e^{-2\pi i k w},$$

where a > 0, t and w are real, $f \in L^2(R)$.

It follows that $\Phi(f) \in L^2$ (Q) and Φ is a linear unitary transformation from $L^2(R)$ onto L^2 (Q).

Moreover an inversion formula is also given by

$f(t) = \int_0^1 (Z f)(t, w) dw, -\infty < t < \infty,$

$\hat{f}(-2\pi w) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-2\pi iwt} (Z f)(t, w) dt$ and

$f(2\pi x) = \frac{1}{\sqrt{2\pi}} \int_0^1 e^{-2\pi ixt} (Z \hat{f})(t, x) dt,$

where $\hat{f}$ is Fourier transform of f [12].

BOHEMIAN SPACE

We recall the L^2 – Boehmian space B_R^2 , see [1].

As in the context of Boehmians in [3] we take the complex vector space as $L^2(R)$ and the commutative semi-group as D(R) with usual convolution defined by

$(\phi * \psi)(x) = \int_{-\infty}^{\infty} \phi(x - t) \psi(t) dt, x \in R,$

and the operation from $L^2(R)$ x D(R) into $L^2(R)$ also as the same convolution functions on R defined above. We take Δ as the set of all sequences (ϕ_n) whose elements from D(R) satisfying

Δ_1 $\int_{-\infty}^{\infty} \phi_n(x) dx = 1$ $\forall n \in N$

Δ_2 $\int_{-\infty}^{\infty} |\phi_n(x)| dx \leq M$ $\forall n \in N$ for some M > 0

Δ_3 $S(\phi_n) \to 0$ as $n \to \infty$, where $S(\phi_n) = \sup\{|x|: x \in R, \phi_n(x) \neq 0\}$.

Now we construct a new Boehmian space as follows:

Take the vector space as $L^2(Q)$ and the commutative semigroup as D(R).

Define $\otimes$: $L^2(Q)$ x D(R) $\to L^2(Q)$ by

$$(F \otimes \phi)(t, w) = \int_{-\infty}^{\infty} F(t, w - x) \phi(x) dx.$$

Lemma 3.1: If $F \in L^2(Q)$ and $\phi \in D(R)$ then $F \otimes \phi \in L^2(Q)$.

Proof: If $F \in L^2(Q) \Rightarrow F: Q \to \mathbb{C}$ such that $\int_0^1 \int_0^1 |F(t, w)|^2 dt\, dw < \infty$,

Where Q = [0, 1) x [0, 1).

We have to prove that $F \otimes \phi \in L^2(Q)$ for $\phi \in D(R)$.

Consider $\int_0^1 \int_0^1 |(F \otimes \phi)(t, w)|^2 dt\, dw = \int_0^1 \int_0^1 |\int_{-\infty}^{\infty} F(t, w - x). \phi(x) dx|^2 dt\, dw$

$$\leq \int_0^1 \int_0^1 (\int_{-\infty}^{\infty} |F(t, w - x)| |\phi(x)| dx)^2 dt\, dw$$

Since R has finite measure with respect to the measure $|\phi(x)| dx$, now we can apply Jensen's inequality and get the last integral dominated by

$$\int_0^1 \int_0^1 \int_{-\infty}^{\infty} |F(t, w - x)|^2 |\phi(x)| dx\, dt\, dw$$

$$= \int_{-\infty}^{\infty} |\phi(x)| dx \int_0^1 \int_0^1 |F(t, w - x)|^2 dt\, dw, \text{ (By Fubini's Theorem)}$$

$$= \int_{-\infty}^{\infty} |\phi(x)| dx \, . ||F||^2 = M.||F||^2 < \infty, \quad \text{(since } F \in L^2(Q) \text{ and } \Delta_2)$$

This implies that, $F \otimes \phi \in L^2(Q)$.

Lemma 3.2: If $\phi_1, \phi_2 \in D$ then $\phi_1 * \phi_2 = \phi_2 * \phi_1 \in D$.

For this Boehmian space also we take the 'Delta Sequences' as Δ.

Lemma 3.3: If $F \in L^2(Q)$, $(\phi_n) \in \Delta$, then $F \otimes \phi_n \to F$ as $n \to \infty$.

Proof: Let $\in > 0$ be given. Using the fact that $C_c(Q)$ is dense in $L^2(Q)$ we can choose $H \in C_c(Q)$ such that

$$||F - H||_2 < \in \quad (1)$$

Now

$$||F \otimes \phi_n - F||_2 \le ||F \otimes \phi_n - H \otimes \phi_n||_2 + ||H \otimes \phi_n - H||_2 + ||H - F||_2 \quad (2)$$

Consider

$$||F \otimes \phi_n - H \otimes \phi_n||_2^2 = \int_0^1 \int_0^1 |(F \otimes \phi_n - H \otimes \phi_n)(t, w)|^2 \, dt \, dw$$

$$= \int_0^1 \int_0^1 | \int_{-\infty}^{\infty} (F(t, w - x) - H(t, w - x)) \, \phi_n(x) \, dx|^2 \, dt \, dw$$

$$\le \int_0^1 \int_0^1 (\int_{-\infty}^{\infty} |F(t, w - x) - H(t, w - x)| . |\phi_n(x)| \, dx)^2 \, dt \, dw$$

$$\le \int_0^1 \int_0^1 \int_{-\infty}^{\infty} |F(t, w - x) - H(t, w - x)|^2 \, |\phi_n(x)| dx \, dt \, dw$$

(By Jensen's inequality)

$$\le \int_{-\infty}^{\infty} |\phi_n(x)| \, dx \, . \int_0^1 \int_0^1 |(F(t, w - x) - H(t, w - x)|^2 \, dt \, dw$$

(By Fubini's Theorem)

$$\le \int_{-\infty}^{\infty} |\phi_n(x)| \, dx \, . ||F - H||^2 \le M \, ||F - H||^2 < M \in^2 \to 0 \text{ as } n \to \infty.$$

Therefore, $||F \otimes \phi_n - H \otimes \phi_n||_2 \to 0$ as $n \to \infty$ (3)

Next

$$||H \otimes \phi_n - H||_2^2 = \int_0^1 \int_0^1 |(H \otimes \phi_n - H)(t, w)|^2 \, dt \, dw$$

$$= \int_0^1 \int_0^1 | \int_{-\infty}^{\infty} (H(t, w - x) - H(t, w)) \, \phi_n(x) \, dx|^2 \, dt \, dw, \quad (\text{By } \Delta_1)$$

$$\le \int_0^1 \int_0^1 (\int_{-\infty}^{\infty} |H(t, w - x) - H(t, w)| \, |\phi_n(x)| dx)^2 \, dt \, dw$$

$$\le \int_0^1 \int_0^1 \int_{-\infty}^{\infty} |H(t, w - x) - H(t, w)|^2 |\phi_n(x)| dx \, dt \, dw$$

(By Jensen's inequality)

$$\le \int_{-\infty}^{\infty} |\phi_n(x)| \, dx \, . \int_0^1 \int_0^1 |(H(t, w - x) - H(t, w)|^2 \, dt \, dw$$

(By Fubini's Theorem)

$$\le \int_{-\infty}^{\infty} |\phi_n(x)| \, dx \, . \iint_k |(H(t, w - x) - H(t, w))|^2 \, dt \, dw \; \forall \, n \ge N \quad (4),$$

Where K compact subset of Q, with support of $H \subset K \subset Q$ and $N_1 \in N$ is such that $S(\phi_n) < 1 \;\; \forall \, n \ge N_1$.

Since H is uniformly continuous on compact set K. Hence for given $\in > 0$ there exists $\delta > 0$ such that

$|H(x, y) - H(u, v)| < \in$ whenever $|(x, y) - (u, v)| < \delta$.

Now choose $N_2 \in N$ such that $S(\phi_n) < \delta \quad \forall\, n \geq N_2$.

Therefore R.H.S. of (4) $< M \in^2 \iint_k \, dt\, dw \leq C \in^2$, for some $0 < C < \infty \to 0$ as $n \to \infty$.

Therefore, $||H \otimes \phi_n - H|| \to 0$ as $n \to \infty$ (5)

Using (1), (3) and (5) in (2), we get, $||F \otimes \phi_n - F|| \to 0$ as $n \to \infty \Rightarrow F \otimes \phi_n \to F$ in $L^2(Q)$ as $n \to \infty$.

Lemma 3.4: If $F_1, F_2 \in L^2(Q)$ and $(\phi_n) \in \Delta$, such that $F_1 \otimes \phi_n = F_2 \otimes \phi_n \ \forall\, n \in N$ then $F_1 = F_2$ in $L^2(Q)$.

Proof: As $F_1 \otimes \phi_n = F_2 \otimes \phi_n \quad \forall\, n \in N$

Letting $n \to \infty$, we get $F_1 = F_2$.

Lemma 3.5: If $F_n \to F$ as $n \to \infty$ in $L^2(Q)$ and $\phi \in D(R)$ then $F_n \otimes \phi \to F \otimes \phi$ as $n \to \infty$ in $L^2(Q)$.

Proof: From above

$$|| F_n \otimes \phi - F \otimes \phi||_2^2 = \int_0^1 \int_0^1 | \int_{-\infty}^{\infty} F_n(t, w - x) - F(t, w - x))\phi(x)\, dx|^2 \, dt\, dw$$

$$\leq \int_0^1 \int_0^1 (\int_{-\infty}^{\infty} |F_n(t, w - x) - F(t, w - x)|\, |\phi(x)|\, dx)^2 \, dt\, dw$$

$$\leq \int_0^1 \int_0^1 \int_{-\infty}^{\infty} |F_n(t, w - x) - F(t, w - x)|^2 |\phi(x)|\, dx\, dt\, dw$$

(By Jensen's inequality)

$$\leq \int_{-\infty}^{\infty} |\phi(x)|\, dx \int_0^1 \int_0^1 |F_n(t, w - x) - F(t, w - x)|^2 \, dt\, dw$$

$$\leq \int_{-\infty}^{\infty} |\phi(x)|\, dx \,.\, ||F_n - F||_2^2 \leq C\, ||F_n - F||_2^2 \to 0 \text{ as } n \to \infty,$$

for some suitable constant C.

$|| F_n \otimes \phi - F \otimes \phi|| \to 0$ as $n \to \infty$

Therefore we get $F_n \otimes \phi \to F \otimes \phi$ in $L^2(Q)$ as $n \to \infty$.

Lemma 3.6: If $F_n \to F$ as $n \to \infty$ in $L^2(Q)$ and $(\phi_n) \in \Delta$ then $F_n \otimes \phi_n \to F$ as $n \to \infty$ in $L^2(Q)$.

Proof: Consider $||F_n \otimes \phi_n - F||_2 \leq ||(F_n - F) \otimes \phi_n||_2 + ||F \otimes \phi_n - F||_2$ (1)

Now, $||(F_n - F) \otimes \phi_n||_2^2 = \int_0^1 \int_0^1 |(F_n - F) \otimes \phi_n(t, w)|^2 \, dt\, dw$

$$= \int_0^1 \int_0^1 | \int_{-\infty}^{\infty} (F_n(t, w - x) - F(t, w - x))\, \phi_n(x)\, dx|^2 \, dt\, dw$$

$$\leq \int_0^1 \int_0^1 (\int_{-\infty}^{\infty} |F_n(t, w - x) - F(t, w - x)|.\, |\phi_n(x)|\, dx)^2 \, dt\, dw$$

$$\leq \int_0^1 \int_0^1 \int_{-\infty}^{\infty} |F_n(t, w - x) - F(t, w - x)|^2 \, |\phi_n(x)|\, dx\, dt$$

(By Jensen's inequality)

$$= \int_{-\infty}^{\infty} |\phi_n(x)|\, dx \int_0^1 \int_0^1 |F_n(t, w - x) - F(t, w - x)|^2 \, dt\, dw$$

$$= \int_{-\infty}^{\infty} |\phi_n(x)|\, dx \,.\, ||F_n - F||_2^2 \;\leq\; M \,.\, ||F_n - F||_2^2 \to 0 \text{ as } n \to \infty$$

(By Δ_2 and $F_n \to F$ as $n \to \infty$)

$$\Rightarrow ||(F_n - F) \otimes \phi_n||_2 \to 0 \text{ as } n \to \infty \qquad (2)$$

Also by Lemma 3.3 we get $||F \otimes \phi_n - F||_2 \to 0$ as $n \to \infty$ (3)

Using (2) and (3) in (1) we get, $||F_n \otimes \phi_n - F||_2 \to 0$ as $n \to \infty$ therefore $F_n \otimes \phi_n \to F$ as $n \to \infty$ in $L^2(Q)$.

Now the Boehmian space B_Q^2 can be constructed in a canonical way. The notation of δ - convergence on B_Q^2 is defined as follows $y_n \overset{\delta}{\to} y$ as $n \to \infty$ if there exists $(\phi_k) \in \Delta$ such that $y_n \otimes \phi_k$ and $y \otimes \phi_k \in L^2(Q)$ and

$y_n \otimes \phi_k \to y \otimes \phi_k$ in $L^2(Q)$.

We shall use the following lemma; whose proof can be taken from [5].

Lemma 3.7: $y_n \overset{\delta}{\to} y$ as $n \to \infty$ in B_Q^2 if and only if there exists $F_{n_k}, F_k \in L^2(Q)$, and $(\phi_k) \in \Delta$, such that $y_n = \left[\frac{F_{n_k}}{\phi_k}\right]$, $y = \left[\frac{F_k}{\phi_k}\right]$ and $F_{n_k} \to F_k$ in $L^2(Q)$ as $n \to \infty$, for each $k \in N$.

Where $\left[\frac{F_{n_k}}{\phi_k}\right]$ and $\left[\frac{Fk}{\phi k}\right]$ denotes equivalence classes containing the quotients of sequences $\frac{F_{n_k}}{\phi_k}$ *and* $\frac{F_k}{\phi_k}$ *respectively.*

ZAK TRANSFORM

Definition 4.1: Define the Zak transform $Z_a : B_R^2 \to B_Q^2$ by

$Z_a\left(\left[\frac{f_n}{\phi_n}\right]\right) \triangleq \left[\frac{\Phi(f_n)}{\phi_n}\right]$, where $\Phi(f_n)$ is a Zak transform of f_n and $\left[\frac{\Phi(f_n)}{\phi_n}\right]$ *denote* the equivalence class containing the quotient of sequence $\frac{\Phi(f_n)}{\phi_n}$.

Lemma 4.1: If $f \in L^2(R)$ and $\phi \in D(R)$ then $\Phi(f * \phi)(t, w) = (\Phi f \otimes \phi)(t, w)$

Proof: $\Phi(f * \phi)(t, w) = \sqrt{a} \sum_{k=-\infty}^{\infty} (f * \phi)(at + ak)\, e^{-2\pi ikw}$

$$= \sqrt{a} \sum_{k=-\infty}^{\infty} \int_{-\infty}^{\infty} f[a(t - x) + ak]\, \phi(x)\, dx\, e^{-2\pi ikw}$$

$$= \int_{-\infty}^{\infty} [\sqrt{a} \sum_{k=-\infty}^{\infty} f(a(t - x) + ak)\, e^{-2\pi ikw}]\, \phi(x)\, dx \quad (\text{Since } \phi \in D(R))$$

$$= \int_{-\infty}^{\infty} \Phi f(t-x, w)\, \phi(x)\, dx$$

$$= (\Phi f \otimes \phi)(t, w)$$

Therefore $\Phi(f * \phi)(t, w) = (\Phi f \otimes \phi)(t, w) \quad \forall (t, w) \in Q$.

Lemma 4.2: The Zak transform $Z_a : B_R^2 \to B_Q^2$ is well defined.

Proof: Let $\left[\frac{f_n}{\phi_n}\right] \in B_R^2$ then $f_n \in L^2(R)$ and $(\phi_n) \in \Delta$. This implies that $\Phi(f_n) \in L^2(Q)$.

We shall show that $\frac{\Phi(f_n)}{\phi_n}$ is a quotient. Since $\left[\frac{f_n}{\phi_n}\right] \in B_R^2 \Rightarrow \frac{f_n}{\phi_n}$ is a quotient we have, $f_n * \phi_m = f_m * \phi_n \;\forall m, n \in$ N.

Applying the classical Zak transform on both sides and by Lemma 4.1 we get

$\Phi(f_n)\otimes \phi_m = \Phi(f_m)\otimes \phi_n \;\; \forall$ m, n $\in$ N $\Rightarrow \frac{\Phi(f_n)}{\phi_n}$ is a quotient.

Next we show that the definition of Z_a is independent of the choice of the representative.

Let $\frac{f_n}{\phi_n} = \frac{g_n}{\psi_n}$, then we have $f_n * \psi_m = g_m * \phi_n \;\forall$ m, n $\in$ N.

Again by applying classical Zak transform and by using Lemma 4.1, we get

$\Phi(f_n) \otimes \psi_m = \Phi(g_m)\otimes \phi_n \;\forall$ m, n $\in$ N. Hence the lemma follows.

Theorem 4.1: The Zak transform $Z_a : B_R^2 \rightarrow B_Q^2$ is a linear map.

Proof: Let $\left[\frac{f_n}{\phi_n}\right], \left[\frac{g_n}{\psi_n}\right] \in B_{R,}^2$ α, β∈ ₵ Now $Z_a \left(\propto \left[\frac{f_n}{\phi_n}\right] + \beta \left[\frac{g_n}{\psi_n}\right]\right) = Z_a\left(\left[\frac{\propto f_n * \psi_n + \beta g_n * \phi_n}{\phi_n * \psi_n}\right]\right)$

$$= \left[\frac{\Phi(\propto f_n * \psi_n + \beta g_n * \phi_n}{\phi_n * \psi_n}\right], \text{ (By definition)}$$

$$= \left[\frac{\propto\Phi(f_n)\otimes\psi_n + \beta\Phi(g_n)\otimes\phi_n}{\phi_n * \psi_n}\right], \quad \text{(By Lemma 4.1)}$$

$$= \left[\frac{\propto\Phi(f_n)}{\phi_n} + \frac{\beta\Phi(g_n)}{\psi_n}\right]$$

$$= \propto \left[\frac{\Phi(f_n)}{\phi_n}\right] + \beta\left[\frac{\Phi(g_n)}{\psi_n}\right]$$

$$= \propto Z_a\left(\left[\frac{f_n}{\phi_n}\right]\right) + \beta\, Za\left(\left[\frac{g_n}{\psi_n}\right]\right)$$

In this proof we have used the fact that Φ is linear wherever it is required.

Theorem 4.2: The Zak transform Z_a:$B_R^2 \rightarrow B_Q^2$ is one-one.

Proof: Let $\left[\frac{f_n}{\phi_n}\right],\left[\frac{g_n}{\psi_n}\right] \in B_R^2$. If $Z_a\left(\left[\frac{f_n}{\phi_n}\right]\right) = Z_a\left(\left[\frac{g_n}{\psi_n}\right]\right)$ then we have $\left[\frac{\Phi(f_n)}{\phi_n}\right] = \left[\frac{\Phi(g_n)}{\psi_n}\right]$ and hence we get

$\Phi(f_n)\otimes \psi_m = \Phi(g_m) \otimes \phi_n \;\forall$ m, n $\in$ N.

Using Lemma 4.1 We get $\Phi(f_n * \psi_m) = \Phi(g_m * \phi_n) \quad \forall$ m, n $\in$ N.

Since, Φ is one-to-one we get, $f_n * \psi_m = g_m * \phi_n \quad \forall$ m, n $\in$ N

(Since Zak transform is isometric from L^2(R) onto L^2 (Q)).

This implies that $\left[\frac{f_n}{\phi_n}\right] = \left[\frac{g_n}{\psi_n}\right]$. Therefore, Z_a is one-to-one.

Theorem 4.3: $\left[\frac{F_n}{\phi_n}\right] = Z_a\left(\left[\frac{f_n}{\phi_n}\right]\right)$ if and only if $F_n \in L^2$ (Q) ∀ n ∈ N, in particular Z_a: $B_R^2 \rightarrow B_Q^2$ is onto.

Proof: Since $\left[\frac{F_n}{\phi_n}\right] = Z_a\left(\left[\frac{f_n}{\phi_n}\right]\right) = \left[\frac{\Phi(f_n)}{\phi_n}\right]$, this implies that $F_n \in L^2$ (Q) ∀ n ∈ N.

Conversely if there exists $\left[\frac{F_n}{\phi_n}\right] \in B_Q^2$ such that $F_n \in L^2(Q)$, then we can choose $f_n \in L^2$(R) Such that $\Phi(f_n) = F_n$ ∀ n ∈ N.

First we show that $\frac{f_n}{\phi_n}$ is a quotient. Since $\frac{F_n}{\phi_n}$ is a quotient we have

$F_n \otimes \phi_m = F_m \otimes \phi_n \;\forall\, m$, n ∈ N i.e. $\Phi(f_n)\otimes \phi_m = \Phi(f_m)\otimes \phi_n \;\forall$ m, n ∈ N.

By Lemma 4.1, we have $\Phi\,(f_n * \phi_m) = \Phi(f_m * \phi_n)\ \forall$ m, n ∈ N

Since Φ is one-one we get, $f_n * \phi_m = f_m * \phi_n\ \forall$ m, n ∈ N⇒ $\frac{f_n}{\phi_n}$ is a quotient.

Therefore $\left[\frac{f_n}{\phi_n}\right] \in B_R^2$, Such that $Z_a\left(\left[\frac{f_n}{\phi_n}\right]\right) = \left[\frac{\Phi(f_n)}{\phi_n}\right] = \left[\frac{F_n}{\phi_n}\right]$

Hence the theorem follows.

Theorem 4.4: The Zak transform Z_a: $B_R^2 \longrightarrow B_Q^2$ is consistent with Φ : L^2(R) ⟶ L^2 (Q).

Proof: Let f ∈ L^2(R). Then by Lemma 4.1, we have

$$Z_a\left(\left[\frac{f * \phi_n}{\phi_n}\right]\right) = \left[\frac{\Phi(f * \phi_n)}{\phi_n}\right] = \left[\frac{\Phi(f)\otimes\phi_n}{\phi_n}\right]$$

Hence the theorem proof is completed.

Theorem 4.5: The Zak transform Z_a: $B_R^2 \longrightarrow B_Q^2$ is consistent with respect to the δ - convergence.

Proof: Let $x_n \overset{\delta}{\to} x$ as n ⟶ ∞ in B_R^2, then by Lemma 3.7 there exist, f_{n_k}, f_k ∈ L^2(R), and (ϕ_n) ∈ Δ such that $x_n = \left[\frac{f_{n_k}}{\phi_k}\right]$, $x = \left[\frac{f_k}{\phi_k}\right]$ and $f_{n_k} \to f_k$ as n → ∞ in L^2(R) for each k ∈ N. Since Φ is continuous from L^2(R) ⟶ L^2 (Q), this implies $\Phi(f_{n_k})$ ⟶ $\Phi(f_k)$ as n ⟶ ∞ in L^2 (Q) for each k ∈ N.

This shows that $\left[\frac{\Phi(f_{n_k})}{\phi_k}\right] \overset{\delta}{\to} \left[\frac{\Phi(f_k)}{\phi_k}\right]$ as n ⟶ ∞ in B_Q^2.

Therefore Z_a is consistent with respect to the δ – convergence.

A COMPARATIVE STUDY

We know that L^2(R) is properly contained in B_R^2 (for example compactly supported distributions belong to B_R^2 but do not represent L^2 – function) and we have proved that Z_a is consistent with Φ. Thus our theory extends the classical Zak transform on L^2(R) to a Boehmian space as a continuous linear map of B_R^2 into B_R^2 whose image can be completely characterized.

CONCLUSIONS

In this paper, we extend the definition of classical Zak transform to larger class and this is consistent with the classical definition. There are some functions which are not Zak transformable in classical sense but for which we can obtain the generalized Zak transform.

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